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STAGNATION FLOW AND HEAT TRANSFER IN A ZONAL MAGNETIC FIELD

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SUMMARY

The problem stated in the title is studied for small values of the diffusivity ratio ϵ and the magnetic force coefficient β, the magnetic field being of internal origin. Uniformly valid expansions are derived for the velocity and magnetic fields in the fluid. It is found that as $\beta \longrightarrow 1$, the viscous layer is brought to rest and the current in the layer is uniform and normal to the wall.

The heat transfer is next calculated at a uniformly heated wall on the assumption of small temperature variations. It is deduced that when $\beta \log(\epsilon^{-1})$ approaches a certain value depending on the wall temperature etc., the thermal boundary layer separates at the stagnation point and, if dissipation is neglected, along the whole wall.

1. *Introduction*

The term 'stagnation flow' refers to the steady viscous flow in the neighbourhood of the forward stagnation point 0 of a bluntnosed body facing a uniform stream. Such a flow is usually treated by approximating the actual body surface about 0 by the tangent plane(or wall) at 0 so that the governing equations may accommodate similarity variables. The classical solutions in the absence of a magnetic field are those of Hiemenz and Homann [1].

MHD extensions of these flows depend on the configuration of the magnetic field and among recent studies (in the subsonic regime) are those of Ludford [2] and Gribben [3]. Such studies, though largely motivated by their relative mathematical simplicity, are also of engineering interest: one area of interest is in the performance of a diagnostic probe inserted in a MHD flow to detect changes in pressure, velocity, etc. Another is in the possibility of using electromagnetic means for controlling skin friction and heat transfer in high speed flight.

Gribben considered two stagnation flows, one plane and the other axisymmetric, in which the magnetic field is parallel to the wall. The solution for the plane study is very simple: in dimensionless form, the velocity and the pressure gradient normal to the wall are identical with those in the Hiemenz flow and the magnetic field follows by a straightforward integration; physical expressions are given by a suitable adjustment of constants.

The axisymmetric flow which Gribben describes in detail was first treated by Axford [4] in a general way. The associated magnetic field which is purely zonal is taken to be generated within the body but it does not vanish at infinity and indeed currents persists there; yet the distant flow is assumed to be undisturbed. It seems doubtful whether such a flow can be realized physically. Gribben derived a perturbation solution for the velocity and magnetic fields for small ϵ, the ratio of momentum to magnetic diffusivity, and later [5] he used an iterative method. The situation is re-examined here under more realistic conditions: the axial current density has a known constant value at the wall and the distant flow is current free, as is proper for an internally generated field. The latter condition also ensures that the magnetic and electric fields vanish at infinity. The flow and field geometry is sketched in the figure below.

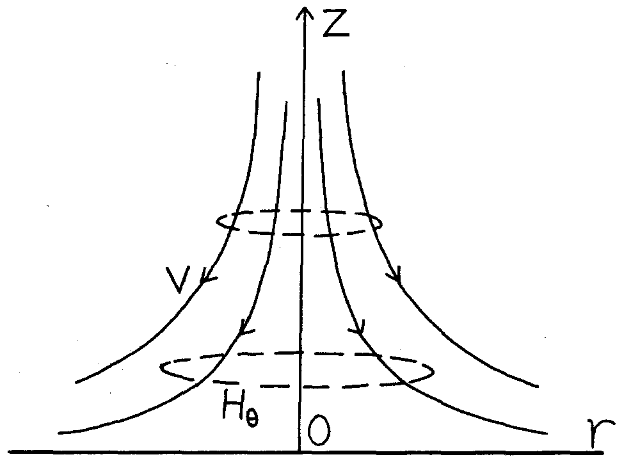

Fig. 1. Streamlines and magnetic field lines.

The last part of this study is concerned with the temperature distribution due to a uniformly heated wall, taking dissipative effects into account. The numerical results derived for the wall heat flux refer to mercury and liquid sodium.

2. *The velocity and magnetic fields*

In terms of cylindrical coordinates (r, θ, z), the velocity is $\mathbf{v} = (v_r, 0, v_z)$ and the magnetic fields is $\mathbf{H} = (0, H_\theta, 0)$. Sufficiently far from the wall $(z = 0)$, we assume a potential flow.

$$\mathbf{v} \sim (ar, 0, -2a(z - z_0)) \tag{1}$$

in which a and z_0 are constants. The first component of (1) will serve as a boundary condition on the velocity. Physically, z_0 stands for the displacement of the inviscid flow by the action of viscous and magnetic forces and is to be determined as part of the solution. Dimensionless variables may now be introduced:

$$\xi = (a/\lambda)^{\frac{1}{2}} z, \quad \beta = \mu A^2/\rho a^2, \tag{2a}$$

$$v_r = raF'(\xi), \quad H_\theta = rAK(\xi), \quad v_z = -2(a\lambda)^{\frac{1}{2}} F(\xi). \tag{2b}$$

The constant A is taken as known and equals half the value of the axial current at the wall, i.e. $j_z(0) = 2A$. The parameter β, a magnetic force coefficient, measures the effect of the imposed current on the flow; $\lambda = (\sigma\mu)^{-1}$ is the magnetic diffusivity. The remaining symbols are standard. Clearly the Homann flow is distorted by the field only if A is non-zero; this in turn implies a conducting wall surface but, with A given, it is unnecessary to prescribe the surface conductivity separately.

On writing the total pressure in the form

$$p + \tfrac{1}{2}\mu H_\theta^2 = -\tfrac{1}{2}\rho a^2 r^2 - 2\rho\lambda a P(\xi), \tag{3a}$$

the z component of the momentum equation may be integrated to yield

$$P(\xi) = F^2 + \epsilon F' + \text{constant.} \tag{3b}$$

The remaining MHD equations reduce to

$$\epsilon F''' + 2FF'' - F'^2 + 1 = \beta K^2,$$
$$K'' + 2FK' = 0, \tag{4}$$

with $F(0) = F'(0) = 0$, $F'(\infty) = 1$, $K(0) = 1$, $K(\infty) = 0$.

The solution is restricted to the case $\epsilon \ll 1$ which is relevant to nearly all non-astrophysical applications. We may write $\epsilon = Rm/Re$, where $Rm = ar_0^2/\lambda$ and $Re = ar_0^2/\nu$ are the magnetic and ordinary Reynolds numbers based on a reference length r_0 such that ar_0^2 is of unit order. (For example, r_0 may be the nose radius of the actual body). Re is usually large in practice and as equations (4) are exact and not boundary layer approximations, Rm need only be $0(1)$ but may be larger provided that $Rm \ll Re$. Equations (4) define a singular perturbation problem: since some surface conditions must be dropped in solving them, they are the outer equations and are valid in what may be called a magnetic region where the vorticity and radial current density are $0(Rm)^{\frac{1}{2}}$, becoming $o(1)$ at the edge of the potential region. Inner variables appropriate to a viscous layer in which the vorticity is $0(Re)^{\frac{1}{2}}$ and the radial current density is $0(Rm)^{\frac{1}{2}}$ are

$$\eta = (a/\nu)^{\frac{1}{2}} z, \quad f(\eta) = \epsilon^{-\frac{1}{2}} F(\xi), \quad k(\eta) = K(\xi). \tag{5}$$

The corresponding inner equations and conditions are

$$f''' + 2ff'' - f'^2 + 1 = \beta k^2,$$
$$k'' + 2\epsilon fk' = 0, \tag{6}$$

and $k(0) = 1$, $f(0) = f'(0) = 0$. The remaining conditions on (6) as well as inner conditions on (4) are to be derived by a match of the two solutions.

Note that when Rm is large so that $Re \gg Rm \gg 1$, the magnetic region contracts into a narrow layer of thickness $0(\lambda/a)^{\frac{1}{2}}$; the viscous layer is now a sublayer of thickness $0(\nu/a)^{\frac{1}{2}}$ and the two together form the MHD boundary layer. It may be shown from (3) that the total pressure variation across the whole layer is $0(Rm)^{-1}$; that across the sublayer is $0(Re)^{-1}$ as in ordinary boundary layer theory.

The solution is assumed to be of the form $F = F_0 + \epsilon^{\frac{1}{2}} F_1$ etc. That for k_0 is simply

$$k_0 = 1 \tag{7}$$

and so, to this order, the axial current is constant in the viscous layer and there is no radial current. Inspection of (5) and (7) shows that the conditions $F_0(0) = 0$ and $K_0(0) = 1$ are necessary; the first equation of (4) then gives

$$f_0'(\infty) = F_0'(0) = (1 - \beta)^{\frac{1}{2}}. \tag{8}$$

The solution for f_0 is

$$f_0 = \alpha h(\alpha\eta) \tag{9}$$

where $\alpha = (1-\beta)^{\frac{1}{4}}$ and h is the Homann function satisfying $h''' + 2hh' - h'^2 + 1 = 0$ and $h(0) = h'(0) = 0$, $h'(\infty) = 1$. The function is tabulated in [1]. Clearly a solution exists only for $\beta < 1$. If $\beta = 1$ there is no flow in the inner layer, all velocity and current changes taking place outside it. This situation, a form of boundary layer separation, is of a type met by Glauert [6] in his study of the flow past a magnetized plate.

The basic outer equations for F_0 and K_0 must be solved numerically in general; this will not be attempted here. Instead we restrict β to be small enough to permit sub-expansions $F_0 = F_{00} + \beta F_{01} + \ldots$, $F_1 = F_{10} + \beta F_{11} + \ldots$, etc. and similarly for the other quantities. Since in any case $\beta < 1$, this restriction is not too severe. (It will turn out in the next section that, with heat transfer present, β must in fact be small.)

The first set of solutions is

$$F_{00} = \xi, \quad K_{00} = \operatorname{erfc}\xi \tag{10}$$

where

$$\operatorname{erfc}\xi = 2\pi^{-\frac{1}{2}} \int_{\xi}^{\infty} e^{-t^2} dt.$$

It might be thought that since $\beta = A = 0$ in this approximation, K_{00} should also vanish. Now $K = j_z/2A$ and it is the physical variable j_z which vanishes with A so that K_{00} represents the inviscid limit of $j_z/2A$ as A tends to zero. From (9) we note that

$$f_{00} = h(\eta), \quad f_{01} = -\tfrac{1}{4}(h + \eta h') \tag{11a}$$

and the outer expansions are

$$f_{00} \sim (\eta - c), \quad f_{01} \sim -\tfrac{1}{4}(2\eta - c) \tag{11b}$$

where $c = 0.5689$. The equation for F_{01} is

$$\xi F''_{01} - F'_{01} = \tfrac{1}{2}\operatorname{erfc}^2\xi \tag{12a}$$

with $F_{01}(0) = F'_{01}(\infty) = 0$. The solution is

$$F_{01} = \tfrac{1}{2}\pi^{-\frac{1}{2}}\Big[2\xi^2 I(\xi) - \pi^{\frac{1}{2}}\xi\operatorname{erfc}^2\xi + e^{-\xi^2}\operatorname{erfc}\xi - \frac{1}{\sqrt{2}}\operatorname{erfc}(\xi\sqrt{2}) + \left(\frac{1}{\sqrt{2}} - 1\right)\Big] \tag{12b}$$

$$\text{where} \quad I(\xi) = \int_{\xi}^{\infty} \frac{e^{-t^2}\operatorname{erfc} t}{t}\, dt \quad (\xi > 0). \tag{13a}$$

For small ξ,

$$I(\xi) \sim -\log\xi - \tfrac{1}{2}\gamma - \log(1+\sqrt{2}) + 2\pi^{-\frac{1}{2}}\xi + 0(\xi^2) \tag{13b}$$

where $\gamma = 0.5772$ is Euler's constant and

$$F_{01} \sim -\tfrac{1}{2}\xi - \pi^{-\frac{1}{2}}\xi^2\log\xi + 0(\xi^2). \tag{14}$$

Equation (14) shows that to $0(\beta)$ the vorticity which is purely zonal and proportional to F'' is logarithmically infinite at the wall in the inviscid flow. A similar result appeared in Ludford's study [2] of plane stagnation flow. Inspection of (14) also shows that a $\beta\epsilon^{\frac{1}{2}}\log(\epsilon^{-1})$ term is required in the f series in order to match with the outer solution. This term, f_L say, is governed by the equation

$$f'''_L + 2hf''_L - 2h'f'_L + 2h''f_L = 0. \tag{15a}$$

The asymptotic form for large η is $f'_L \sim b(\eta - c)$ and b is to be determined by matching with the outer function F_{11} of the next stage. The complete expression for F_{11} is not required; the inner expansion of F'_{11} may be shown to be of the form

$$F'_{11} \sim 2\pi^{-\frac{1}{2}} c \log \xi + \text{const.}$$

and a match with the logarithmic term then gives

$$f'_L \sim \pi^{-\frac{1}{2}}(\eta - c) \quad \text{as} \quad \eta \longrightarrow \infty. \tag{15b}$$

A numerical solution of (15a) with (15b) and the usual inner conditions gives $f''_L(0) = 0.3662$. Equation (14) shows that F'_{01} contains a $\xi \log \xi$ term. Written in inner variables, the coefficient of the part $\eta \epsilon^{\frac{1}{2}} \log(\epsilon^{-1})$ agrees with that of η in (15b). The ξ term of F_{01} in (14) and the η term of f_{01} in (11b) are also seen to match.

The second order solution for the velocity in the absence of a magnetic field is simply $F_{10} = -c$, which leads to the matching condition $f'_{10}(\infty) = 0$. The function f_{10} then satisfies a homogeneous problem and hence $f_{10} \equiv 0$.

The next function K_{01} determines the $0(\beta)$ effect of the flow on the current distribution. The governing equation is

$$K''_{01} + 2\xi K'_{01} = 4\pi^{-\frac{1}{2}} F_{01} \; e^{-\xi^2} \tag{16a}$$

where F_{01} is given by (12); the conditions are $K_{01}(0) = K_{01}(\infty) = 0$. A first integral is

$$K'_{01} = \frac{e^{-\xi^2}}{3\pi} \Big[-4\xi^3 I(\xi) - \tfrac{1}{2}\pi^{\frac{1}{2}}(6\xi^2 + 1)\operatorname{erfc}^2\xi + 4\xi e^{-\xi^2}\operatorname{erfc}\xi - 3\sqrt{2}\,\xi \operatorname{erfc}(\xi\sqrt{2}) + \pi^{-\frac{1}{2}} e^{-2\xi^2} - 3(2-\sqrt{2})\xi + C \Big] \tag{16b}$$

where $I(\xi)$ is defined by (13a). Integration of (16b) from 0 to ∞ gives

$$C = \frac{\pi}{3} + 1 - \frac{1}{\sqrt{3}} + 2 \log \frac{\sqrt{2}+1}{\sqrt{3}-1} = 3.8564.$$

The inner expansion may now be deduced to be

$$K_{01} \sim 0.3750\xi + 0(\xi^3). \tag{16c}$$

Logarithmic terms do not appear in the k series until the $\beta\epsilon^2\log(\epsilon^{-1})$ stage and will not concern us here. The earlier terms, k_1 and k_2, which are the $0(\epsilon^{\frac{1}{2}})$ and $0(\epsilon)$ perturbations of the axial current in the viscous layer are both of the form $k_i = a_i\eta$ $(i = 1, 2)$, and matching requires that $a_i = K_{i-1}(0)$. Recalling that $a_i = a_{i0} + \beta a_{i1} + \ldots$, we have $a_{10} = -2\pi^{-\frac{1}{2}} = -1.1284$ and $a_{11} = 0.3750$. The value of a_{20} depends on the equation

$$K''_{10} + 2\xi K'_{10} = -4\pi^{-\frac{1}{2}} c \; e^{-\xi^2} \tag{17a}$$

with the usual condition $K_{10}(\infty) = 0$ and a match with k_{10} gives $K_{10}(0) = 0$. The solution is

$$K_{10} = 2\pi^{-\frac{1}{2}} c \; (e^{-\xi^2} - \operatorname{erfc}\xi) \tag{17b}$$

and hence $a_{20} = K'_{10}(0) = 4c/\pi = 0.7243$. The functions k_{10}, k_{11} and k_{20}

are now determined.

The solution will not be developed further and we proceed to summarize the results obtained so far. As ϵ is quite small (the value 10^{-6} being typical) we must have $\beta \gg \epsilon$ if magnetic forces are to affect the flow sensibly and it will be assumed that $\beta \geqslant 0(\epsilon^{\frac{1}{2}})$.

The magnetic and velocity field components have the following expansions to indicated orders in β and ϵ, uniformly valid in $z \geqslant 0$:

$$H_\theta(r,z) = r A\left[\operatorname{erfc}\xi + \beta K_{01}(\xi) + \epsilon^{\frac{1}{2}}K_{10}(\xi)\right] \tag{18}$$

$$v_z(z) = -2(a\lambda)^{\frac{1}{2}}\left[\beta F_{01}(\xi) + \epsilon^{\frac{1}{2}}h(\eta)\right] \tag{19a}$$

$$v_r(r,z) = ra\left[h'(\eta) + \beta\left\{f'_{01}(\eta) + F'_{01}(\xi) + \tfrac{1}{2}\right\} + \beta\epsilon^{\frac{1}{2}}\log(\epsilon^{-1})f'_L(\eta)\right] \tag{19b}$$

where f_{01}, F_{01}, f_L, K_{01}, K_{10} are given by (11), (12), (15), (16), (17). The corresponding expressions for the currents $j_z = 2H_\theta/r$, $j_r = -\frac{1}{2}Arj'_z$ follow from (18). Within the boundary layer, in particular, $j_z = 2A$ and

$$j_r(r) = Ar(a/\lambda)^{\frac{1}{2}}\ (1.1284 - 0.3750\beta - 0.7243\epsilon^{\frac{1}{2}} + \ldots) \tag{20}$$

The current paths in the fluid are the curves $r^2 j_z$ = const., j_z being a composite expansion; the paths start at $r = \infty$ in the magnetic region, gradually enter into the boundary layer and into the wall and return to $r = \infty$. The total radial current outflow from the region $z \geqslant 0$, $z \leqslant R$ say, is $2\pi AR^2$.

The volume flow defect out of $r \leqslant R$, $z \geqslant 0$ is $Q = 2\pi aR^2 z_0$ where $z_0 = \int_0^\infty (1 - v_r/ar)dz$ is the displacement constant or thickness occurring in (1). Using (12) and recalling that $F_{10} = -c$ we have

$$z_0 = (a/\lambda)^{-\frac{1}{2}}(0.0826\beta + 0.5689\epsilon^{\frac{1}{2}} + \ldots). \tag{21}$$

When $Rm = ar_0^2/\lambda$ is $0(1)$ (see remarks below (4)) and $\beta = 0(\epsilon^{\frac{1}{2}})$, the displacement effect due to magnetic and viscous forces is comparable and additive but at a smaller Rm and for $\beta \geqslant 0(\epsilon^{\frac{1}{2}})$, (21) shows that a relatively extensive magnetic region can be accommodated between the wall and the potential region.

The last quantity of interest is the wall stress. There is no Maxwell component; the skin friction τ may be computed from (9) and (15). The result for small β is

$$\tau = \rho r(\nu a^3)^{\frac{1}{2}}\left[1.3119(1-\beta)^{\frac{3}{4}} + 0.3662\beta\epsilon^{\frac{1}{2}}\log(\epsilon^{-1}) + \ldots\right]. \tag{22}$$

The foregoing analysis pre-supposes a conducting wall and a narviscous layer along it. We recall that the solution cannot be developed for $\beta > 1$. As $\beta \longrightarrow 1$, the field becomes strong enough to arrest the flow in this layer: in this stagnant fluid the current is uniform and axial, and the magnetic force is balanced by the radial pressure gradient.

3. *Heat transfer from an isothermal wall*

Let the wall surface $z = 0$ be maintained at a uniform temperature T_w and let $T_\infty (< T_w)$ be the temperature of the external flow. Then provided that $(T_w - T_\infty)/T_\infty$ and the external Mach number are small, the energy and momentum equations are uncoupled. For liquids (with which we are concerned here) the Mach condition is irrelevant but small relative temperature differences are necessary in order to preserve constancy of properties, especially the viscosity. Since the expansion coefficient is small, being of the order of 10^{-4} per degree C for mercury at 15°C and liquid sodium at 200°C, the pressure term may be dropped from the energy equation which in steady flow becomes

$$\rho c_p (\mathbf{v}.\nabla)T = k_T \nabla^2 T + \Phi + \mathbf{j}^2/\sigma \tag{23}$$

where T is the liquid temperature, c_p is the specific heat, k_T the thermal conductivity and the last two terms represent viscous and ohmic dissipation respectively.

For conducting liquids the Prandtl number Pr is rather small (typically about 0.01) but may be taken to be 0(1) compared with the much smaller parameter ϵ. Accordingly the thermal boundary layer will be comparable in thickness with the viscous layer (though actually thicker by about $Pr^{-\frac{1}{2}}$ times) and each is only about $\epsilon^{\frac{1}{2}}$ as thick as the inviscid-magnetic region referred to in section 2. In this region thermal conduction and viscous dissipation are negligible, being $0(\epsilon)$, but not ohmic dissipation which is $0(\beta)$ and we have assumed that $\beta \geqslant 0(\epsilon^{\frac{1}{2}})$. Hence to a first approximation the heat balance outside the viscous layer is between convection and ohmic dissipation so that (23) reduces to

$$\rho c_p (\mathbf{v}_0 . \nabla)T = \mathbf{j}_0^2/\sigma \tag{24}$$

the suffix o signifying that outer variables of a basic solution are being referred to. Since the external temperature is not attained at the boundary layer edge but at the outer edge, the condition proper to (24) is $T = T_\infty$ at $z = \infty$ and it is sufficient. Outer conditions for the boundary layer approximation of (23) are to be deduced from the inner expansion of (24), the remaining condition being the prescribed wall temperature.

A suitable dimensionless form for T is

$$T - T_\infty = (T_w - T_\infty)L_0(\xi) + \frac{a^2 r^2}{c_p} M_0(\xi). \tag{25}$$

Substituting in (24) gives

$$F_0 L_0' + \beta W K_0^2 = 0, \tag{26a}$$

$$F_0 M_0' - F_0' M_0 + \tfrac{1}{2}\beta K_0'^2 = 0 \tag{26b}$$

with $L_0(\infty) = M_0(\infty) = 0$. The functions F_0 and K_0 representing the axial velocity and current in the inviscid flow have appeared earlier. The parameter $W = 2a\lambda/c_p(T_w - T_\infty)$, of the form of an Eckert number, is taken to be 0(1).

Note that non-trivial solutions for L_0, M_0 are due to the ohmic forcing terms in (26). Expanding $L_0 = L_{00} + \beta L_{01}$ etc., we have that $L_{00} = M_{00} = 0$ and

$$L_{01} = W \int_\xi^\infty \frac{\operatorname{erfc}^2 t}{t}\, dt \quad (\xi > 0). \tag{27a}$$

The small ξ expansion is

$$L_{01} \sim -W \left[\log\xi + d + 0(\xi\log\xi)\right] \tag{27b}$$

where $d = \frac{1}{2}\gamma + \log 4 + 4\pi^{-\frac{1}{2}} \int_0^{\pi/4} \log\cos t\, dt = 1.5649$. On the other hand (26b) shows that M_{01} is finite at the origin, in fact $M_{01}(0) = 2/\pi$.

In terms of inner variables $l_0(\eta) = L_0(\xi)$, $m_0(\eta) = M_0(\xi)$ where $\eta = (a/\nu)^{\frac{1}{2}} z$ as before, equation (23) reduces to the thermal boundary layer equations:

$$Pr^{-1}l_0'' + 2f_0 l_0' + 2\beta W = 0, \tag{28a}$$

$$Pr^{-1}m_0'' + 2f_0 m_0' - 2f_0' m_0 + f_0''^2 + \beta k_1'^2 = 0 \tag{28b}$$

with $l_0(0) = 1$, $m_0(0) = 0$; f_0 is given by (9) and k_1 is referred to below (16c). The basic variables l_{00}, m_{00} satisfy (28) with $\beta = 0$ and f_0 replaced by f_{00} or h. The boundary conditions are $l_{00}(0) = 1$, $l_{00}(\infty) = m_{00}(0) = m_{00}(\infty) = 0$. The dimensionless wall heat flux components in the absence of a magnetic field are $l_{00}'(0)$ and $m_{00}'(0)$, the latter being the contribution from viscous dissipation. The computed values at Pr = 0.025(mercury), 0.0075 (liquid sodium) are shown in the Table below.

Equation (27b) shows that a logarithmic stage, $\beta\log(\epsilon^{-1})$, is already present in the l_0 series and the expansions are $l_0 = l_{00} + \beta\log(\epsilon^{-1})l_L + \beta l_{01} + \ldots$, $m_0 = m_{00} + \beta m_{01} + \ldots$ The equation for l_L is

$$l_L'' + 2Pr\, h l_L' = 0 \tag{29a}$$

with $l_L(0) = 0$ and the matching condition $l_L(\infty) = \frac{1}{2}W$ deduced from (27b). We find that

$$l_L'(0) = -\tfrac{1}{2}W\theta \tag{29b}$$

where $\theta = l_{00}'(0)$, a function of Pr. Next we have

$$\tfrac{1}{2}Pr^{-1}l_{01}'' + h l_{01}' = -(f_{01} l_{00}' + W) \tag{30a}$$

with $l_{01}(0) = 0$, $l_{01} \sim -W(d + \log\eta)$ as $\eta \to \infty$; h, f_{01} are given by (11). It can be shown that

$$l_{01}'(0) = \theta(qW - \tfrac{1}{4}) \tag{30b}$$

where $q = d + \lim_{\eta\to\infty}\left[\log\eta - 2Pr\int_0^\eta X(t)\int_t^\infty X^{-1}(u)\,du\,dt\right]$ and $X(t) = \exp(2Pr\int_0^t h(u)du)$. The function m_{01} is given by

$$\tfrac{1}{2}Pr^{-1}m_{01}'' + h m_{01}' - h' m_{01} = -(f_{01}m_{00}' - f_{01}' m_{00} + h'' f_{01}'' + 2/\pi) \tag{31}$$

with $m_{01}(0) = 0$ and $m_{01}(\infty) = 2/\pi$. The results $k_{10} = -2\pi^{-\frac{1}{2}}$ and $M_{01}(0) = 2/\pi$ have been used here. Equations (30) and (31) were integrated numerically and the results are tabulated below.

TABLE

Pr	$\theta = l_{00}'(0)$	$l_L'(0)/\theta$	$l_{01}'(0)/\theta$	$m_{00}'(0)$	$m_{01}'(0)$
0.025	-0.1627	$-\frac{1}{2}W$	$2.55W - \frac{1}{4}$	0.0182	0.461
0.0075	-0.0925	$-\frac{1}{2}W$	$3.15W - \frac{1}{4}$	0.0057	0.272

The heat transfer at the wall is given by

$$(\nu/a)^{\frac{1}{2}}\frac{\partial T}{\partial z}(r,0) = (T_w - T_\infty)\theta\left[1 - \beta\left\{\tfrac{1}{2}W\log(\epsilon^{-1}) - qW + \tfrac{1}{4}\right\}\right] + \frac{a^2r^2}{c_p}\left\{m_{00}'(0) + \beta\, m_{01}'(0)\right\}. \tag{32}$$

The Table shows that the second curly bracket in (32) is positive so that the applied field increases the dissipative contribution to the heat transfer for $r > 0$. Next, since $\epsilon = 10^{-7}$ for mercury and 10^{-5} for sodium, the first curly bracket also remains positive for all W and we conclude that when $\beta\log(\epsilon^{-1}) < 0(1)$ the applied field decreases the heat transfer at the stagnation point 0 itself and, provided that $a^2r^2/c_p(T_w - T_\infty) \leq 0(1)$, also on a disk of radius r centred at 0.

As β increases with the strength of the applied field, $\beta\log(\epsilon^{-1}) = s$ say, will become 0(1) and the l_0 series will fail to be asymptotic. Physically we can expect s to attain a critical value at which the thermal layer is disrupted, in the sense that the heat flux vanishes at the stagnation point and, if dissipation is ignored, along the entire wall. An analogous situation with the velocity boundary layer arose in [6]. When s is 0(1) the relevant parameter is $\bar{\epsilon} = 1/\log(\epsilon^{-1})$ since β is then $0(\bar{\epsilon})$, and to find the critical value of s we seek a series solution of (26a) and (28a) in terms of $\bar{\epsilon}$. In general a smooth development would require a modified set of equations but here this is unnecessary and indeed compatibility with the solution of Section 2 and the temperature condition at the wall will not permit a change of variables. The dominant term in the $l_0'(0)$ expansion $(1 - \frac{1}{2}sW)$ and hence the flux vanishes at 0 when $s = 2/W$. Alternatively, consider formal series of the type $l_0(\eta, s) = \bar{l}_{00}(\eta, s) + \bar{\epsilon}\bar{l}_{01}(\eta, s)$ etc. Then $\bar{L}_{00} = 0$ as before; $\bar{l}_{00}$ satisfies the same equation and surface condition as l_{00} but we must insist on the outer condition $\bar{l}_{00}(\infty) = \alpha_0$ where α_0 is an 0(1) parameter depending on s. It is easily shown that $\bar{l}_{00}'(0, s) = (1 - \alpha_0)\theta$ and hence the flux at 0 remains negative provided that $\alpha_0 < 1$. Guided by our earlier solution (27b) we deduce that $\bar{L}_{01} \sim -sW(d + \log\xi) = -sW(d + \log\eta - \frac{1}{2}\bar{\epsilon}^{-1})$ in inner variables. The first two terms are the outer expansion of $\bar{l}_{01}$ but the last term remains unmatched if α_0 is zero. Matching $\bar{l}_{00}$ with $\bar{L}_{00} + \bar{\epsilon}\bar{L}_{01}$ gives

$$\alpha_0 = 0 + \bar{\epsilon}(\tfrac{1}{2}sW\bar{\epsilon}^{-1}) \text{ and hence } s = 2\alpha_0/W < 2/W.$$

The field strenght is therefore limited by the inequality

$$\beta < \frac{c_p(T_w - T_\infty)}{a\lambda\log(\epsilon^{-1})}.$$

As the latter value is reached, the thermal layer separates i.e. the heat transfer is reduced to zero at the stagnation point and, if dissipation is neglected, on the entire wall surface.

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